A One-Dimensional Treatment of Inviscid Jet or Duct Flow

Philip A. Thompson*
Rensselaer Polytechnic Institute, Troy, N. Y.

Approximate equations involving one space dimension are derived for variable-area jet or duct flow. At a given cross section, the radial velocity and the sound speed are represented by distributions satisfying necessary boundary conditions for continuity and momentum. By integrating the full equations of inviscid motion over the cross section, quasi-one-dimensional equations with time and axial distance as independent variables are obtained. The equations for duct flow are the same as the characteristic equations from a purely one-dimensional treatment, but with an additional term. The equations for jet flow are also in characteristic form. Sample solutions for four steady jet flows are compared with available results from experiment and with the method of characteristics; agreement is satisfactory. Possible extensions of the technique employed are discussed.

THE one-dimensional representation of flow in a variable area duct is well known and yields, in many cases, surprisingly accurate results. In such a treatment, average values are used to represent quantities (axial velocity, pressure, etc.) which actually vary over a given cross section. In this paper, the equations of motion will be integrated over the cross section for assumed distributions of the dependent variables. This leads to a new and simplified set of equations in which radially variable quantities are represented by average or boundary parameters. This approach is familiar in boundary-layer studies.[1]

The flow to be considered is either axisymmetric or plane-symmetric: the space coordinates are r and z, applicable to both symmetries.† The flow may vary with time t. A definable boundary $r_*(z, t)$ is assumed to exist.

For inviscid flow, the equations of motion, integrated from $r = 0$ to $r = r_*$, are

$$\int_0^{r_*}\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial r}\right)dr = 0 \tag{1}$$

$$\int_0^{r_*}\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial z}\right)dr = 0 \tag{2}$$

$$\int_0^{r_*}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{v}{\rho}\frac{\partial \rho}{\partial r} + \frac{u}{\rho}\frac{\partial \rho}{\partial z} + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} + N\frac{v}{r}\right)dr = 0 \tag{3}$$

$$N = \begin{cases} 0 & \text{plane flow} \\ 1 & \text{axisymmetric flow}\end{cases}$$

where v and u are the radial and axial velocity components, respectively.‡ The energy equation will be replaced by the homentropic condition, s = const, so that there is in effect only one thermodynamic variable. The state equations in the following will be appropriate to an ideal gas with constant specific heats. The emphasis here is on unsteady motion, and the equations will be developed for that case.

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* Associate Professor of Mechanical Engineering; now with Fluid Mechanics Research Group, Large Steam Turbine-Generator Division, General Electric Company, Schenectady, N. Y. Member AIAA.

† For conciseness, only the terms "radial" and "axial" will be used. The respective meanings are "in the r direction" and "in the z direction."

‡ The integrands of Eqs. (1–3) may of course be multiplied by an arbitrary function of r, such as r^n. This function may be considered to be a "weighting factor"; in this paper it is unity.

The component velocities will be assumed of the form

$$u = u(z, t) \tag{4}$$

$$v = (r/r_*)v_*(z, t) \tag{5}$$

where starred quantities represent values at the boundary $r = r_*$. The condition $u = u(z, t)$ means of course that axial velocity is assumed uniform over any given cross section. The form of v satisfies the requirement of continuity on the axis (plane) $r = 0$; this form is somewhat comparable to the linear strain assumption in thin plate theory. It will be necessary to assume one further distribution, namely, for the speed of sound c. This will be specified later.

The integration for radial-momentum (1) is now performed. Making use of the Leibniz rule and (5), the first term becomes

$$\int_0^{r_*}\frac{\partial v}{\partial t}dr = \frac{r_*}{2}\frac{\partial v_*}{\partial t} - \frac{v_*}{2}\frac{\partial r_*}{\partial t}$$

where the time derivatives on the right-hand side imply z held constant. It should be noted that $\partial r_*/\partial t$ is not the same as v_*. The next two terms are evaluated similarly, using (4) and (5)

$$\int_0^{r_*}\left(v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z}\right)dr = \frac{{v_*}^2}{2} + \frac{ur_*}{2}\frac{\partial v_*}{\partial z} - \frac{uv_*}{2}\frac{\partial r_*}{\partial z}$$

where the right-hand z derivatives imply time held constant. The last term is evaluated by using the homentropic condition for an ideal gas

$$dP/\rho = dc^2/(\gamma - 1)$$

where γ is the ratio of specific heats. The integration then gives

$$\int_0^{r_*}\frac{1}{\rho}\frac{\partial P}{\partial r}dr = \frac{{c_*}^2 - {c_0}^2}{\gamma - 1}$$

where c_0 represents the sound speed on the axis $r = 0$. The radial momentum equation may now be written

$$\frac{\partial v_*}{\partial t} + u\frac{\partial v_*}{\partial z} = \frac{2}{\gamma - 1}\frac{{c_0}^2 - {c_*}^2}{r_*} \tag{6}$$

where three terms have been eliminated by using the boundary equation

$$\frac{Dr_*}{Dt} = v_* = \frac{\partial r_*}{\partial t} + u\frac{\partial r_*}{\partial z} \tag{7}$$

The integration for axial-momentum (2) is straightforward, the pressure term being transformed as before so that

$$\int_0^{r_*}\frac{1}{\rho}\frac{\partial P}{\partial z}dr = \frac{1}{\gamma - 1}\int_0^{r_*}\frac{\partial c^2}{\partial z}dr =$$

$$\frac{\partial}{\partial z}\int_0^{r_*}\frac{c^2}{\gamma - 1}dr - \frac{{c_*}^2}{\gamma - 1}\frac{\partial r_*}{\partial z}$$

and the full equation becomes

$$r_* \frac{\partial u}{\partial t} + r_* u \frac{\partial u}{\partial z} + \frac{\partial}{\partial z} \int_0^{r_*} \frac{c^2}{\gamma - 1}\, dr - \frac{c_*{}^2}{\gamma - 1} \frac{\partial r_*}{\partial z} = 0 \qquad (8)$$

The assumption $u = u(z, t)$ has resulted in loss of the term corresponding to radial transport of axial momentum.

The integration for continuity is accomplished with the help of the homentropic transformation

$$\frac{d\rho}{\rho} = \frac{2}{\gamma - 1} \frac{dc}{c} = \frac{2}{\gamma - 1} d(\ln c)$$

The second term in (3) is then

$$\int_0^{r_*} \frac{v}{\rho} \frac{\partial \rho}{\partial r}\, dr = \frac{2}{\gamma - 1} \int_0^{r_*} v \frac{\partial}{\partial r} (\ln c) dr$$

which is integrated by parts to give

$$\frac{2}{\gamma - 1} v_* \ln c_* - \frac{2}{\gamma - 1} \frac{v_*}{r_*} \int_0^{r_*} \ln c\, dr$$

The remaining terms are readily evaluated, making use of the Leibniz rule as needed. The boundary equation (7) is again used to eliminate some terms, and the resulting equation may be written

$$\frac{2}{\gamma - 1} \left\{ \frac{\partial}{\partial t} + u \frac{\partial}{\partial z} - \frac{v_*}{r_*} \right\} \int_0^{r_*} \ln c\, dr + r_* \frac{\partial u}{\partial z} + (N + 1) v_* = 0 \qquad (9)$$

Equations (6–9) are the new equations of motion, involving only z and t. It remains to replace the various integrals of sound speed c by some appropriate average values.

On the axis, $r = 0 = v$. If this condition is substituted into the integrand of Eq. (1), the result immediately obtained is that $\partial P/\partial r$ must vanish at $r = 0$. This requirement is satisfied by a parabolic sound speed distribution

$$c = c_0 + (c_* - c_0)(r/r_*)^2 \qquad (10)$$

The mean value $\bar{c}$ is defined to be

$$\bar{c} = \frac{1}{r_*} \int_0^{r_*} c dr = c_0 + \frac{c_* - c_0}{3} \qquad (11)$$

In general, the change in c across a given section is small, though the corresponding pressure change may be large (e.g., for air $P \sim c^7$). Thus $\delta = |(c_* - c_0)/c_0| \ll 1$ and terms of order δ^2 will be neglected henceforth. The logarithm may be approximated from (10)

$$\ln c \approx \ln c_0 + \frac{c_* - c_0}{c_0} \left(\frac{r}{r_*} \right)^2$$

Using this with Eq. (11)

$$\int_0^{r_*} \ln c\, dr = r_* \ln \bar{c}$$

and the continuity equation (9) becomes

$$\frac{2}{\gamma - 1} \frac{\partial \bar{c}}{\partial t} + \frac{2}{\gamma - 1} u \frac{\partial \bar{c}}{\partial z} + \bar{c} \frac{\partial u}{\partial z} = - \frac{(N + 1) v_* \bar{c}}{r_*} \qquad (12)$$

where the boundary equation (7) has again been used. This is a conventional one-dimensional continuity equation, with an additional "radial leakage" term on the right.§

§ This equation may be derived by considering the conservation of mass for a fixed control volume of thickness dz. An average density $\bar{\rho}$ is used and then eliminated by using $\overline{d\rho}/\bar{\rho} = [2/(\gamma - 1)] d\bar{c}/\bar{c}$. Similar comments apply to the momentum equations.

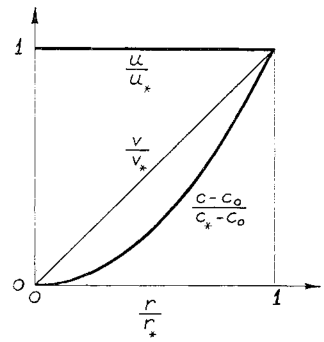

Fig. 1 Radial distribution of sound speed *c*, axial velocity *u*, and radial velocity *v*.

The various distributions are diagrammed in Fig. 1. In general, they have been chosen to have the most simple form consistent with the boundary and axis conditions.

With the preceding assumptions for $c(r)$, the integrals in (6) and (8) become, respectively,¶

$$c_*{}^2 - c_0{}^2 = [3\bar{c} + \tfrac{3}{4}(c_* - \bar{c})][c_* - \bar{c}] \approx 3\bar{c}(c_* - \bar{c})$$

$$\int_0^{r_*} c^2 dr = \frac{r_*}{5} [5\bar{c}^2 - (c_* - \bar{c})^2] \approx r_* \bar{c}^2$$

The first of these is substituted straightaway into the radial momentum equation. The second integral is differentiated with respect to z in the axial momentum equation. Then (6) and (8) become, respectively,¶

$$\frac{\partial v_*}{\partial t} + u \frac{\partial v_*}{\partial z} = \frac{6}{\gamma - 1} \frac{\bar{c}(\bar{c} - c_*)}{r_*} \qquad (13)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{2}{\gamma - 1} \bar{c} \frac{\partial \bar{c}}{\partial z} = - \frac{2}{\gamma - 1} \frac{\bar{c}(\bar{c} - c_*)}{r_*} \frac{\partial r_*}{\partial z} \qquad (14)$$

where a term $(\bar{c} - c_*)^2$ has again been neglected in the later equation. Equation (13) relates outward acceleration at the boundary to central overpressure ($\bar{c} > c_*$), whereas (14) relates axial acceleration to axial pressure gradient and an axial force component exerted on the boundary (right side).

Equations (12–14), together with (7), are now the equations of motion. They contain five unknowns, and so an additional condition is necessary. Two cases are discussed in the following.

Duct Flow

In this case the flow boundary $r_*(z)$ is known and the number of unknowns is correspondingly reduced. The boundary equation (7) becomes simply

$$dr_*/dz = r_*' = v_*/u \qquad (15)$$

and thus the derivatives of v_* can be expressed

$$(\partial v_*/\partial t) = r_*'(\partial u/\partial t)$$
$$(\partial v_*/\partial z) = r_*'(\partial u/\partial z) + u r_*'' \qquad (16)$$

The boundary sound speed c_* is variable; it does not appear, however, in any of the derivative terms of (12–14), so that no differential equation explicitly in c_* can be found. It can however be eliminated between (13) and (14), and making use of (15) and (16), this gives

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{6}{(\gamma - 1)[3 - (r_*')^2]} \bar{c} \frac{\partial \bar{c}}{\partial z} = \frac{u^2 r_*' r_*''}{3 - (r_*')^2} \qquad (17)$$

This reduces to the usual quasi-one-dimensional momentum

¶ Equation (13) is sensitive to the form of the sound speed distribution assumed. For the form $c \sim (r/r_*)^n$, the right-hand coefficient $6/(\gamma - 1)$ becomes in general $[4(n + 1)/n(\gamma - 1)]$.

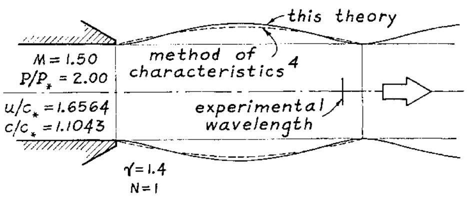

Fig. 2 Supersonic underexpanded axisymmetric flow from a nozzle.

equation for $r_*' = 0$. A more interesting result however is obtained by using the approximation $3-(r_*')^2 \approx 3$, which is valid for ordinary ducts. Then (12) and (17) can be combined to give equations in the well-known characteristic form

$$\left\{\frac{\partial}{\partial t} + (u \pm \bar{c})\frac{\partial}{\partial z}\right\}\left(u \pm \frac{2}{\gamma - 1}\bar{c}\right) = ur_*'\left[\frac{ur_*''}{3} \mp \frac{(N+1)\bar{c}}{r_*}\right] \quad (18)$$

which can be solved by integrating along characteristics. These differ from the usual characteristic forms[2] only in the addition of the r_*'' (wall curvature) term on the right.

For steady flow, the momentum and continuity equations (17) and (12) give a variant of the usual "throat equation"

$$\frac{1}{u}\frac{du}{dz} = \frac{r_*'\{r_*''(M^2/3) + [(N+1)/r_*]\}}{M^2 - 1} \quad (19)$$

where $M = u/\bar{c}$ is the average Mach number. This does not alter the classical one-dimensional arguments about acceleration through $M = 1$ at a throat. It does however introduce the effect of wall curvature from the added term. The ratio of the two terms in braces is thus a rough measure of the departure from one-dimensionality. In a typical example cited by Shapiro,[3] the added r_*'' term and the $(N + 1)/r_*$ term are in the ratio of 1 to 30, although this is at Mach number unity, and the departure increases with M^2.

Jet Flow

Consider now the flow which is bounded by a nominally stationary atmosphere a. If the atmosphere is compliant ($\rho_a c_a \ll \rho_* c_*$), it is appropriate** to take the boundary condition $P(r_*) = P_* =$ const, and correspondingly, $c(r_*) = c_* =$ const. For steady flow, this condition is appropriate without restriction. It is interesting to note that the two types of flow discussed here correspond to extremes in compliance as follows: 1) free boundary (jet flow), $\rho_a c_a \to 0$ $P_* =$ const, and 2) fixed boundary (duct flow), $\rho_a c_a \to \infty$ $(\partial r_*/\partial t) = 0$; whereas any "real" flow lies somewhere between these limits.

For the free boundary ($c_* =$ const) situation, which is now assumed, the system of four equations has only four unknowns. Equations (12) and (14) are added and subtracted to give the characteristic form for jet flow:

$$\left\{\frac{\partial}{\partial t} + (u \pm \bar{c})\frac{\partial}{\partial z}\right\}\left(u \pm \frac{2}{\gamma - 1}\bar{c}\right) = -\frac{2}{\gamma - 1}\frac{\bar{c}(\bar{c} - c_*)}{r_*}\frac{\partial r_*}{\partial z} \mp \frac{(N+1)v_*\bar{c}}{r_*} \quad (20)$$

Equations (7) and (13) are also necessary. This system of equations may be solved by integrating along characteristics, $(dz/dt) = u \pm \bar{c}, u$.

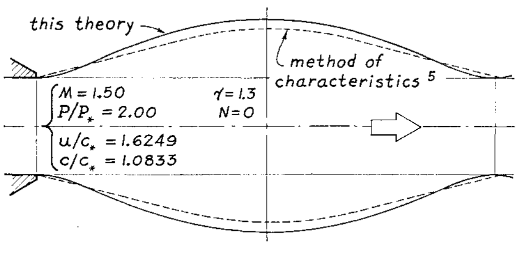

Fig. 3 Supersonic underexpanded plane flow from a nozzle.

** This would be the case for example if a jet of cold nitrogen moves through warm helium; or for an ultrapressure water jet in the atmosphere (the equations given here are appropriate to a compressible liquid described by a Tait equation).

Sample Results for Steady Jet Flow

For comparison with other methods, some calculations have been made for steady jet flow. Actually, if only steady flow is of interest, some refinement can be made by considering the radial variation of axial velocity (see later under discussion). This was not done in the equations presented here, since axial velocity was assumed independent of radius.

For steady flow, the foregoing relations (20, 7, and 13), give

$$\frac{du}{dz} = \frac{v_*\bar{c}}{r_*(u^2 - \bar{c}^2)}\left[(N+1)\bar{c} - \frac{2}{\gamma - 1}(\bar{c} - c_*)\right] \quad (21)$$

$$\frac{d\bar{c}}{dz} = \frac{v_*\bar{c}}{r_*(u^2 - \bar{c}^2)}\left[-\frac{\gamma - 1}{2}(N+1)u + \frac{\bar{c}(\bar{c} - c_*)}{u}\right] \quad (22)$$

$$dr_*/dz = v_*/u \quad (23)$$

$$\frac{dv_*}{dz} = \frac{6}{\gamma - 1}\frac{\bar{c}(\bar{c} - c_*)}{ur_*} \quad (24)$$

which may be solved, for given initial data, by a numerical "marching forward" scheme. The first equation gives the satisfactory result that subsonic and supersonic flows display opposite behavior with respect to acceleration, and the second equation likewise with respect to change of mean sound speed. In the accompanying figures are shown some comparisons with method of characteristics calculations[4, 5] and an experimental result.[6]

In the underexpanded jet solutions (Figs. 2 and 3), the jet boundary cannot show the discontinuity in slope (e.g., at the lip of the nozzle) given by the method of characteristics solution, since v_* changes only continuously in Eq. (24). Correspondingly, the pressure distribution at the nozzle discharge is a step in the actual case, whereas this method assumes a roughly parabolic distribution. In general, the internal velocity and pressure distributions cannot be correct in detail, although qualitatively they are correct. For example, in Fig. 3 the central pressure at maximum expansion is 0.45 atm from the method of characteristics, whereas this theory has 0.32 atm. This kind of behavior is to be expected from an integral-averaging method, which should however give reasonable results for the gross behavior of the flow. In this connection, it is interesting to note that the wavelength of the jet is predicted almost exactly.

Figure 4 shows the boundary computed by this method compared to the boundary obtained from a shadowgraph

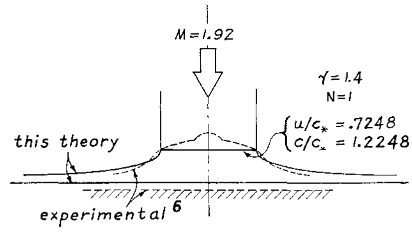

Fig. 4 Subsonic axisymmetric stagnation flow aft of a jet shock.

of a stagnating jet. The cup-like dotted lines in the center of the jet show the experimental shock structure, which in this treatment is necessarily replaced by a plane shock. The downstream flow is purely subsonic. Figure 5 shows the boundary of a diverging jet.

This method has not been applied to incompressible jets, for which the sound speed is not an appropriate variable. One can, however, let γ and c approach infinity, corresponding to decreasing compressibility. Then (21) becomes

$$(du/dz) = -(N + 1)(v_*/r_*)$$

which with (23) is just the continuity equation for a steady incompressible jet.

It is interesting to note that Eqs. (21–24) are invariant under the transformations $z \rightarrow -z, v_* \rightarrow -v_*$. This implies that a solution (such as that in Fig. 2) that passes through $v = 0$ will "retrace," that is, show symmetry about $z(v_* = 0)$.

The agreement with known steady flow solutions is encouraging and suggests that consideration of unsteady cases would be of interest.

Discussion

The assumed form $u(z, t)$ for axial velocity is somewhat unsatisfactory. The integrand of the axial-momentum equation (2) is clearly not satisfied at a point, since substitution of the assumed distributions leads to an equation of the type $f(z, t) = g(r, z, t)$. It is unfortunately not possible in general to describe the variation $u(r)$ by, say, two parameters, as in the case of c, since this would result in an excess of unknowns in the resulting equations.

In the case of unsteady flow, any assumed distribution may not be representative, the form of the actual distribution being variable with time. By integrating over the flow cross section, however, the gross effect of such inconsistencies is diminished.

If one is interested only in steady flow, the adiabatic energy equation

$$u^2 + v^2 + \frac{2}{\gamma - 1} c^2 = \frac{2}{\gamma - 1} c_{st}^2 = \text{const} \qquad (25)$$

gives additional information for the assignment of radial distributions. Specifically, the axial velocity u can now be considered to vary with radius.

An additional condition, not used here, is given by the irrotationality of the flow,

$$\nabla \times \mathbf{u} = 0 \qquad (26)$$

This can be used in conjunction with (25). Some preliminary work by the author gives reason to hope that a more accurate

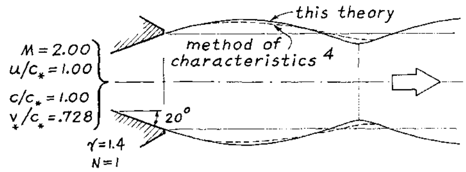

Fig. 5 Supersonic correctly expanded axisymmetric flow from a diverging nozzle.

calculation, of the general type presented here, is thereby possible

It will be interesting to see whether the derived equations (21–24) satisfy the foregoing energy equation in an "average" sense. Differentiating (24) and replacing u, v, c by average values u, $\bar{v} = v_*/3^{1/2}$, $\bar{c}$, one finds that Eqs. (20–23) satisfy the "average" differential equation exactly. Similarly, an average continuity equation, $\bar{\rho}uA = \text{const}$, is satisfied identically.

It is highly unlikely that the methods given here would yield results for steady flow comparable to those from the method of characteristics. On the other hand, they are also valid for subsonic flow and probably represent an improvement over ordinary one-dimensional theory [e.g., Eqs. (18)] for unsteady flow by yielding a "second-order" correction and provide the only means for dealing with jet flows from a one-dimensional point of view. The technique of integrating over the cross section with consistent distributions is simple enough and can be extended to other types of bounded flows.

Finally, the unsteady flow of a jet through an atmosphere which does not have negligible acoustic impedance ($\rho_a c_a \neq 0$) can be treated by relating the boundary pressure and acceleration by means of piston theory. For example, in the case of a plane jet ($N = 0$), the boundary sound speed is approximately, with simple waves,

$$c_* = c_*^0 + [(\gamma_a - 1)/2](\partial r_*/\partial t)$$

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